The basics of chapter 6: (6.1, 6.2, 6.7) Norm, orthogonality and inner Products

or "length, perpendicularity and the dot product"

Key idea: The dot product of two vectors in $\mathbb{R}^n$ generalizes to what we call an inner product on a vector space. This allows us to extend the key ideas of length and perpendicularity in $\mathbb{R}^2$ to <u>any</u> vector space.

We define the dot product of two vectors in $\mathbb{R}^n$: $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

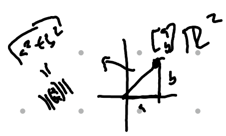
$$\vec{u} \bullet \vec{v} = \vec{u}^T \vec{v} = [u_1\ u_2 \cdots u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

$\boxed{\text{algebraic properties}}$

Ex $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^4$ $\rightsquigarrow \vec{u} \bullet \vec{v} = \cdots = 2.$

Def: The norm (or length) of $\vec{v}$ in $\mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v} \bullet \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

(A unit vector $\vec{u}$ is a vector of norm 1: $\|\vec{u}\| = 1$.) if time, normalize $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \rightsquigarrow \|\vec{u}\| = \sqrt{5}.$

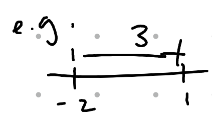


$\begin{bmatrix} a \\ b \end{bmatrix} \bullet \begin{bmatrix} a \\ b \end{bmatrix} = a^2 + b^2 \rightsquigarrow \left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\| = \sqrt{a^2+b^2}.$ ✓

- ~~picture, e.g. 3 dimension vector~~
- ~~distance [illegible]~~
- ⇒ orthogonality, phytagoran theorem and e.g. $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, maybe start here.
- now, orthogonal sets, bases, orthonormal bases and more these helps us do a few useful ideas ~ GS, least squares
- general <u>inner products</u> definition, general norm, general orthogonality.
- <u>ex</u>

Ex Find $\|\vec{v}\|$ and a unit vector with the same direction of $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Recall that in $\mathbb{R}$, the distance between two numbers a, b is $|a - b|$



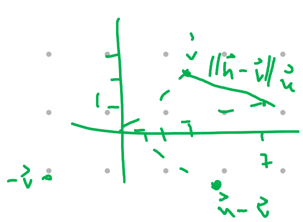
$|-2 - 1| = 3$

We can use the norm to define a similar notion of distance in $\mathbb{R}^n$

Def: For $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, the distance between $\vec{u}$ and $\vec{v}$ is given by

$$dist(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

Ex Compute the distance between $\vec{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

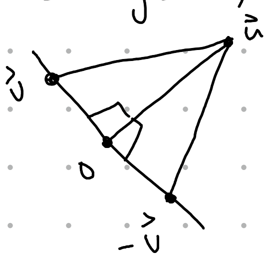


Ex What is the distance of $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^6$

$\vec{u} - \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ $= \sqrt{\quad 14}.$

We now move to generalizing perpendicularity.

Orthogonality: intuitively, $\vec{u}, \vec{v}$ are orthogonal if $\|\vec{u}-\vec{v}\| = \|\vec{u}+\vec{v}\|$.

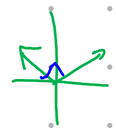


One can show $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 - 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2$
and $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2$

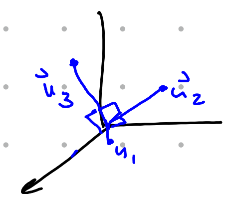
these are equal if $\vec{u}\cdot\vec{v} = 0$. Which yields

Def: Two vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ are orthogonal if $\vec{u}\cdot\vec{v} = \vec{0}$.

Ex| Show $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are orthogonal.



Ex| Show $\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ -7/2 \end{bmatrix}$ are all orthogonal.



In this case, we call $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ an orthogonal set.

If in addition, these vectors are unit, we call the set orthonormal.

Due to time, we cannot discuss these sets further, but note that an orthonormal basis is very nice (as it nearly perfectly emulates the standard basis of $\mathbb{R}^n$). We can always find these by the Gram-Schmidt process.

We now generalize these geometric notions to any vector space by way of an inner product.

Def: An inner product - - - - - - definition

$\rightsquigarrow$ norm, length, unit vector, distance

Ex) On the vector space $C[a,b]$ of continuous functions, we use the integral to define an inner product

$$\langle f, g\rangle = \frac{1}{b-a}\int_a^b f(t)g(t)\,dt.$$

Then $\|f\| = \sqrt{\frac{1}{b-a}\int_a^b [f(t)]^2\,dt}$, $\|f-g\| = \sqrt{\frac{1}{b-a}\int_a^b [f(t)-g(t)]^2\,dt}$

To be specific, work in $C[0,1]$ and consider polynomials

$$p_1(t) = 1, \quad p_2(t) = 2t-1, \quad p_3(t) = 12t^2$$

Compute the norm of $p_3(t)$,
the distance btwn p_1, p_2 and that they're orthogonal.

$$\|p_3(t)\|^2 = \frac{1}{1-0}\int_0^1 (12t^2)(12t^2)\,dt = \int_0^1 24t^4\,dt = \frac{24}{5}t^5\Big|_0^1 = \frac{24}{5}$$

$$\Rightarrow \|p_3(t)\| = \sqrt{24/5}.$$

$$\|p_1-p_2\|^2 = \int_0^1 (2t-2)^2\,dt = \int_0^1 4t^2-8t+4\,dt = \frac{4}{3}t^2-4t^2+4t\Big|_0^1 = \frac{4}{3}$$

$$\Rightarrow \mathrm{dist}(p_1,p_2) = \frac{2}{\sqrt{3}}.$$

$$\langle p_1, p_2\rangle = \int_0^1 (2t-1)(1)\,dt = t^2-t\Big|_0^1 = 0.$$ so orthogonal!

Talk about why this is a good notion of "distance".

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

b. $(\mathbf{u} + \mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

c. $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$

d. $\mathbf{u}\cdot\mathbf{u} \geq 0, \quad$ and $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

An **inner product** on a vector space V is a function that, to each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V, associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ and satisfies the following axioms, for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars c:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0 \quad$ and $\quad \langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$

A vector space with an inner product is called an **inner product space**.